

# AN ELEMENTARY PROOF OF UNIQUENESS OF MARKOFF NUMBERS WHICH ARE PRIME POWERS

YING ZHANG

**ABSTRACT.** We present a very elementary proof of the uniqueness of Markoff numbers which are prime powers or twice prime powers, in the sense that it uses neither algebraic number theory nor hyperbolic geometry.

## 1. INTRODUCTION

**1.1. Markoff numbers.** In his celebrated work on the minima of indefinite binary quadratic forms, A. A. Markoff [13] was naturally led to the study of Diophantine equation—now known as the *Markoff equation*

$$x^2 + y^2 + z^2 = 3xyz. \quad (1)$$

The solution triples  $(x, y, z)$  in positive integers are called by Frobenius [9] the *Markoff triples*, and the individual positive integers occur the *Markoff numbers*.

For convenience, we shall not distinguish a Markoff triple from its permutation class, and when convenient, usually arrange its elements in ascending order. Following Cassels [5], we call the Markoff triples  $(1, 1, 1)$  and  $(1, 1, 2)$  *singular*, while all the others *non-singular*. It is easy to show that the elements of a non-singular Markoff triple are all distinct.

In ascending order of their maximal elements, the first 12 Markoff triples are:

$$(1, 1, 1), (1, 1, 2), (1, 2, 5), (1, 5, 13), (2, 5, 29), (1, 13, 34), (1, 34, 89), \\ (2, 29, 169), (5, 13, 194), (1, 89, 233), (5, 29, 433), (89, 233, 610);$$

while the first 40 Markoff numbers as recorded in [19] are:

$$1, 2, 5, 13, 29, 34, 89, 169, 194, 233, 433, 610, 985, 1325, 1597, 2897, \\ 4181, 5741, 6466, 7561, 9077, 10946, 14701, 28657, 33461, 37666, 43261, \\ 51641, 62210, 75025, 96557, 135137, 195025, 196418, 294685, 426389, \\ 499393, 514229, 646018, 925765.$$

**1.2. Sketch of Markoff's work.** Let  $f(\xi, \eta) = a\xi^2 + b\xi\eta + c\eta^2$  be a binary quadratic form with real coefficients. The discriminant of  $f$  is defined as  $\delta(f) = b^2 - 4ac$ , and the minimum  $m(f)$  of  $f$  is defined as

$$m(f) = \inf |f(\xi, \eta)|,$$

where the infimum is taken over all pairs of integers  $\xi, \eta$  not both zero.

Two quadratic forms  $f(\xi, \eta)$  and  $g(\xi, \eta)$  are said to be *equivalent* if there exist integers  $a, b, c, d$  such that  $ad - bc = \pm 1$  and  $f(a\xi + b\eta, c\xi + d\eta) = g(\xi, \eta)$ .

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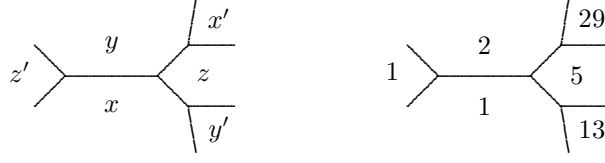


FIGURE 1. Tree structure of a Markoff triple and its neighbors

Then Markoff's aforementioned work on the minima of real indefinite binary quadratic forms can be stated as follows.

**MARKOFF'S THEOREM.** *Let  $f$  be a real indefinite binary quadratic form. Then inequality  $m(f)/\sqrt{\delta(f)} > 1/3$  holds if and only if  $f$  is equivalent to a multiple of a Markoff form.*

Here the *Markoff form* associated to a Markoff triple  $(m, m_1, m_2)$  with  $m \geq m_1 \geq m_2$  is defined as an indefinite binary quadratic form with integer coefficients, as follows. First, let  $u$  be the least non-negative integer such that

$$um_1 \equiv m_2 \pmod{m} \quad \text{or} \quad um_1 \equiv -m_2 \pmod{m}.$$

Since  $0 \equiv m(3m_1m_2 - m) = m_1^2 + m_2^2 \equiv (u^2 + 1)m_1^2 \pmod{m}$  and, as will be shown in §1.5,  $\gcd(m_1, m) = 1$ , we have  $u^2 + 1 \equiv 0 \pmod{m}$ . Now let

$$v = (u^2 + 1)/m.$$

The Markoff form associated to Markoff triple  $(m, m_1, m_2)$  is then defined as

$$\phi_{(m, m_1, m_2)}(\xi, \eta) = m\xi^2 + (3m - 2u)\xi\eta + (v - 3u)\eta^2. \quad (2)$$

Note that for  $\phi := \phi_{(m, m_1, m_2)}$  we have  $\delta(\phi) = 9m^2 - 4$  and  $m(\phi) = m$ .

**Remark.** Note that Markoff [12] [13] used continued fractions to obtain his results, and his proofs were only sketched. Dickson [8, Ch.VII] gave a detailed interpretation of it. Frobenius [9] made a systematic study of the Markoff numbers, based on which Remak [15] presented a proof of Markoff's Theorem using no continued fractions. Markoff's above result also has a well-known equivalent formulation in terms of the approximation of irrationals by rationals; see Cassels [5] and Cusick–Flahive [7] for detailed explanations.

**1.3. Neighbors of a Markoff triple.** That Markoff equation (1) is particularly interesting lies in the fact that it is a quadratic equation in each of the variables, and hence new solutions can be obtained by a simple process from a given one,  $(x, y, z)$ . To see this, keep  $x$  and  $y$  fixed and let  $z'$  be the other root of (1), regarded as a quadratic equation in  $z$ . Rewriting (1) as  $z^2 - 3xyz + (x^2 + y^2) = 0$ , we have  $z + z' = 3xy$  and  $zz' = x^2 + y^2$ . Thus  $z'$  is a positive integer and  $(x, y, z')$  is another solution triple to (1) in positive integers, that is, a Markoff triple. Similarly, we obtain two other Markoff triples  $(x', y, z)$  and  $(x, y', z)$ . We call these three new Markoff triples thus obtained the *neighbors* of the  $(x, y, z)$ . See Figure 1 for an illustration.

**1.4. Reduction.** In [13, pp.397–398], Markoff showed that every Markoff triple can be obtained from the simplest by appropriately iterating the above process.

**THE REDUCTION THEOREM.** *Every Markoff triple can be traced back to  $(1, 1, 1)$  by repeatedly performing the following operation on Markoff triples:*

$$(x, y, z) \mapsto (x, y, z') := (x, y, 3xy - z), \quad (3)$$

where the elements of  $(x, y, z)$  is arranged so that  $x \leq y \leq z$ .

Note that to perform the next operation, one needs to first rearrange the elements of  $(x, y, z')$  in ascending order. As an example, we see

$$\begin{aligned} (13, 194, 7561) &\mapsto (13, 194, 5) \sim (5, 13, 194) \mapsto (5, 13, 1) \sim (1, 5, 13) \\ &\mapsto (1, 5, 2) \sim (1, 2, 5) \mapsto (1, 2, 1) \sim (1, 1, 2) \mapsto (1, 1, 1). \end{aligned}$$

A simple proof of the theorem is given in [5, pp.27–28]; see also [7, pp.17–18]. The idea is that operation (3) reduces the maximal elements of Markoff triples as long as the input triple is non-singular. Indeed, one has  $x < y < z$  and  $(z - y)(z' - y) = zz' - (z + z')y + y^2 = x^2 + 2y^2 - 3xy^2 < 0$ ; hence  $z' < y$ .

Here we give a slightly different proof, the idea of which we get from [2].

**PROOF.** The operation  $(x, y, z) \mapsto (x, y, z')$  reduces the lengths,  $x + y + z$ , of Markoff triples exactly when  $z' < z$ . Therefore, after a finite number of times of length reduction, one stops when  $z' \geq z$ , or equivalently,  $2z \leq 3xy$ . We claim that  $z = 1$  in this case, and hence  $(x, y, z) = (1, 1, 1)$ . Indeed, if  $z \geq 2$  then one obtains from  $x \leq y \leq z$  and  $2z \leq 3xy$  that

$$1 = \frac{x}{3yz} + \frac{y}{3zx} + \frac{z}{3xy} \leq \frac{1}{6} + \frac{1}{3} + \frac{1}{2} = 1. \quad (4)$$

This forces that  $x = y, z = 2$  and  $x = 1, y = z$  both hold, a contradiction.  $\square$

**1.5. First properties of Markoff numbers.** As an immediate corollary of the Reduction Theorem, we see that the elements of a Markoff triple are pairwise coprime. Moreover, since  $zz' = x^2 + y^2$  and  $\gcd(x, y) = 1$ , a Markoff number is not a multiple of 4, and each odd prime factor of a Markoff number is  $\equiv 1 \pmod{4}$ . Consequently, every odd Markoff number is  $\equiv 1 \pmod{4}$  and every even Markoff number is  $\equiv 2 \pmod{8}$ . Indeed, it is shown in [20] that every even Markoff number is  $\equiv 2 \pmod{32}$ .

**1.6. An illustration.** The Reduction Theorem tells that, starting from  $(1, 2, 5)$  and generating new neighbors repeatedly, one will obtain all the Markoff triples. This is depicted as an infinite binary tree in Figure 2 in which all the Markoff numbers appear in the regions while all non-singular Markoff triples appear around vertices. In this shape it seems to be first drawn by Thomas E. Ace on his web-page <http://www.minortriad.com/markoff.html>.

**1.7. The uniqueness problem.** A problem then arises naturally: *Does every Markoff number appear exactly once in the regions in Figure 2?* In other words, are there any repetitions among all the numbers occur?

The following conjecture on the uniqueness of Markoff numbers/triples was first mentioned explicitly by G. Frobenius as a question in his 1913 paper [9]. It asserts that a Markoff triple is uniquely determined by its maximal element. (And we shall simply say that a Markoff number  $z$  is *unique* if the following is true for  $z$ .)

**THE UNICITY CONJECTURE.** *Suppose  $(x, y, z)$  and  $(\tilde{x}, \tilde{y}, z)$  are Markoff triples with  $x \leq y \leq z$  and  $\tilde{x} \leq \tilde{y} \leq z$ . Then  $x = \tilde{x}$  and  $y = \tilde{y}$ .*

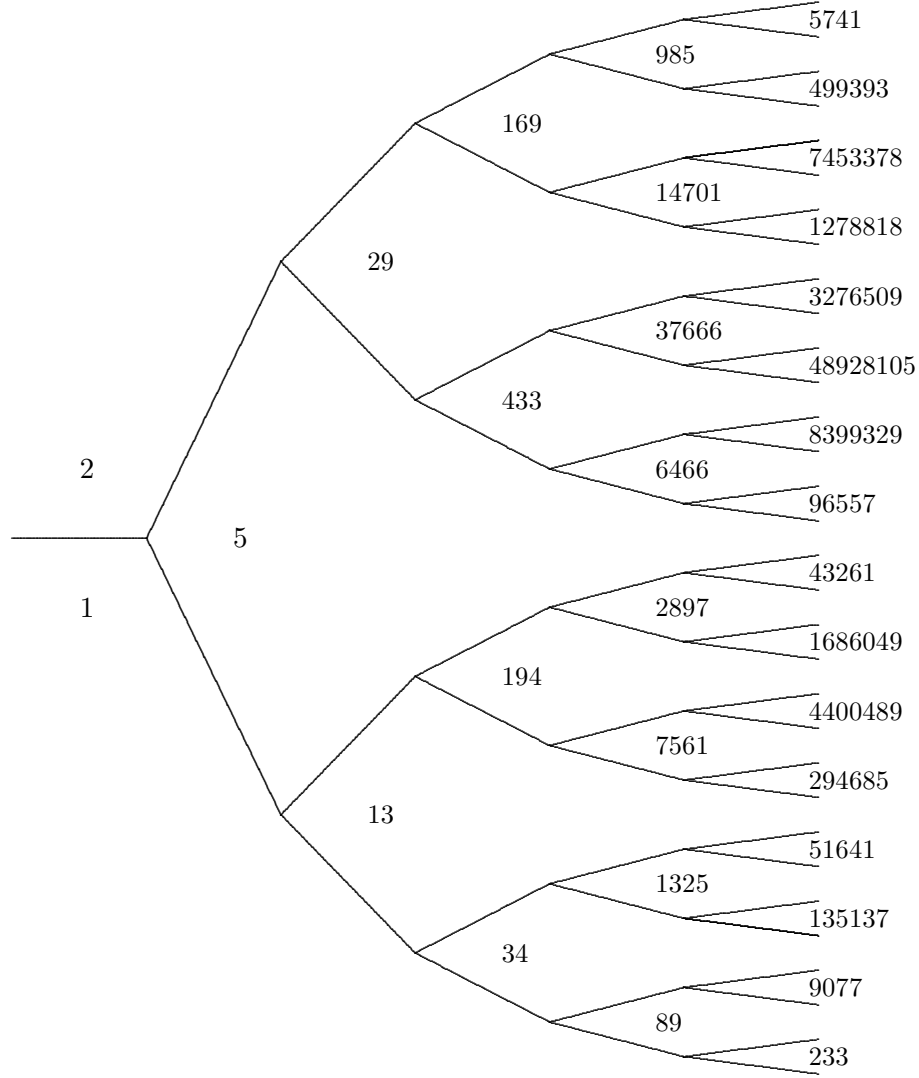


FIGURE 2. Markoff numbers in an infinite binary tree

The conjecture has been proved only for some special subsets of the Markoff numbers. The following affirmative result for Markoff numbers which are prime powers or twice prime powers was first proved independently and partly by Baragar [1], Button [3] and Schmutz [17] using either algebraic number theory ([1],[3]) or hyperbolic geometry ([17]). And a stronger result along the same lines has been obtained later by Button in [4]; in particular, a Markoff number is shown to be unique if it is a “small” ( $\leq 10^{35}$ ) multiple of a prime power.

**Theorem 1** (Baragar [1]; Button [3]; Schmutz [17]). *A Markoff number is unique if it is either a prime power or twice a prime power.*

This paper is motivated by a simple proof of Theorem 1 recently published by Lang and Tan [11], which uses some elementary facts from the hyperbolic geometry of the modular torus with one cusp, as used by Cohn in [6]. The aim of this paper is to present in detail a completely elementary proof of Theorem 1 that uses neither algebraic number theory nor hyperbolic geometry so that an average reader will be able to fully understand it with no difficulty. Though it is later clear that all the needed ingredients of the proof were already known as early as 1913 in Frobenius' work, we must admit that we first obtain them from hyperbolic geometry, especially, that used in [11] and [6].

The rest of the paper is organized as follows. In §2 we parametrize Markoff numbers using non-negative rationals (slopes)  $t \in \hat{\mathbf{Q}} \cap [0, \infty]$ . We also define  $u_t$  as in §1.2 and verify some properties of the pairs  $(m_t, u_t)$ . Then in §3, with the help of a simple lemma (Lemma 4), we give the promised elementary proof of Theorem 1. In §4 we introduce the so-called Markoff matrices to generate all Markoff numbers. Certain properties of these matrices are then discussed in §5. In particular, alternative proofs of Lemmas 2 and 3 will be given. Finally, in §6 we give a geometric explanation of the Markoff numbers and related numbers.

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## 2. SLOPES OF MARKOFF NUMBERS

**2.1. Slopes of Markoff numbers.** It is natural and very useful to associate to each Markoff number a slope, that is, an ordered pair of non-negative coprime integers. This was first done by Frobenius in [9] where he set

$$m(1, 0) = 1, m(0, 1) = 2, m(1, 1) = 5, m(1, 2) = 13, m(2, 1) = 29, \dots$$

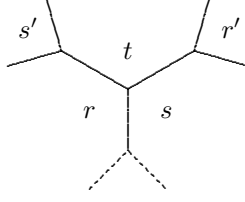
(These pairs are also called by Cusick and Flahive [7] the FROBENIUS COORDINATES of Markoff numbers). Note that, by identifying  $(\mu, \nu)$  with  $\nu/\mu$ , the slopes are nothing but the positive rationals together with 0 and  $\infty$ . In the latter context we shall write  $m_r$  for  $m(\mu, \nu)$  where  $r = \nu/\mu$ .

Let us write  $\hat{\mathbf{Q}} := \mathbf{Q} \cup \{\infty\}$ . We shall also call  $\infty = \frac{1}{0} = \frac{-1}{0}$  a rational. Then the set of slopes we consider is the set of rationals in  $[0, \infty]$ , that is,  $\hat{\mathbf{Q}} \cap [0, \infty]$ .

**2.2. Farey sum of rationals.** There is a simple but useful way to obtain all the positive rationals by making the so-called Farey sums repeatedly. Specifically, starting with  $0 = \frac{0}{1}$  and  $\infty = \frac{1}{0}$  (of level 0), all positive rationals can be generated, level by level, as follows:

$$\begin{aligned} \frac{1}{1} &= \frac{0+1}{1+0}; \\ \frac{1}{2} &= \frac{0+1}{1+1}, \frac{2}{1} = \frac{1+1}{1+0}; \\ \frac{1}{3} &= \frac{0+1}{1+2}, \frac{2}{3} = \frac{1+1}{2+1}, \frac{3}{2} = \frac{1+2}{1+1}, \frac{3}{1} = \frac{2+1}{1+0}; \\ \frac{1}{4} &= \frac{0+1}{1+3}, \frac{2}{5} = \frac{1+1}{3+2}, \frac{3}{5} = \frac{1+2}{2+3}, \frac{3}{4} = \frac{2+1}{3+1}, \frac{4}{3} = \frac{1+3}{1+2}, \frac{5}{3} = \frac{3+2}{2+1}, \frac{5}{2} = \frac{2+3}{1+1}, \frac{4}{1} = \frac{3+1}{1+0}; \end{aligned}$$

and so on. (To obtain the negative rationals, one starts from  $\infty = \frac{-1}{0}$  and  $0 = \frac{0}{1}$  instead and makes the Farey sums recursively as above). In particular, we have the notion of *Farey level* for positive rationals, with levels 1 to 4 shown as above. To

FIGURE 3. Farey triples  $(r, t, s)$ ,  $(r, s', t)$  and  $(t, r', s)$ 

obtain all the rationals in  $[0, \infty]$  of Farey level  $n + 1$ , we simply start with all those of Farey level not exceeding  $n$ , arrange them in ascending order, and make Farey sum for each pair of adjacent ones among them. In particular, we are allowed to prove a proposition concerning all the positive rationals by induction on the Farey levels of the rationals involved. In what follows we shall make the above idea precise and present some basic facts that will be needed in later part of this paper.

By the standard reduced form of a rational number  $t$  we mean the unique fractional expression  $t = \nu/\mu$  where  $\mu, \nu$  are coprime integers with  $\mu \geq 0$ . Two rationals  $r, s$  are said to be *Farey neighbors* (and that they form a *Farey pair*) if they have standard reduced forms  $r = b/a$  and  $s = d/c$  so that  $ad - bc = \pm 1$ . Given a Farey pair  $r, s$  with standard reduced forms  $r = b/a$  and  $s = d/c$ , their *Farey sum* is defined as

$$r \oplus s := \frac{b + d}{a + c} \quad (5)$$

which is certainly in its standard reduced form. (Note that in terms of  $(a, b)$  and  $(c, d)$  regarded as plane vectors, the Farey sum is just the vector sum). Clearly,  $r \oplus s = s \oplus r$ . It is easy to see that  $r \oplus s$  falls in between  $r$  and  $s$  and is a common Farey neighbor of  $r$  and  $s$ . We shall call the ordered triple  $(r, t, s)$  a *Farey triple*.

It follows from the Euclidean algorithm that every positive rational can be written in a unique way as the Farey sum of a Farey pair of rationals in  $[0, \infty]$ . Indeed, for a given positive rational  $t$ , among all its Farey neighbors there are exactly two,  $r$  and  $s$ , having smaller or the same denominators; it can be easily shown that  $r$  and  $s$  form a Farey pair and  $t = r \oplus s$ . We call  $r$  and  $s$  the *direct descents* of  $t$ . As a consequence, it is easy to see that in every Farey pair in  $\hat{\mathbf{Q}} \cap [0, \infty]$ , the one with smaller denominator or numerator has smaller Farey level and is a direct descent of the other. Hence it can be shown by induction that all rationals between  $\frac{0}{1}$  and  $\frac{1}{0}$  will appear in the above process of recursively making Farey sums.

To end this subsection, we give a formal definition of the notion of Farey level. First, we set the Farey level of each of  $\frac{0}{1}$  and  $\frac{1}{0}$  to be 0. Recursively, for a Farey pair  $r, s$  in  $\hat{\mathbf{Q}} \cap [0, \infty]$ , we define the Farey level of their Farey sum  $t = r \oplus s$  to be the sum of their respective Farey levels. In this way we then have recursively defined a Farey level for each  $t \in \hat{\mathbf{Q}} \cap [0, \infty]$ .

**2.3. Further properties of Markoff numbers.** It is easy to see that, in terms of slopes, each Markoff triple is then of the (more natural) form  $(m_r, m_t, m_s)$  where  $(r, t, s)$  is a Farey triple in  $\hat{\mathbf{Q}} \cap [0, \infty]$  with  $r < t < s$ .

We define for each  $t \in \hat{\mathbf{Q}} \cap [0, \infty]$  an integer  $u_t$  with  $0 \leq u_t \leq m_t$  as follows. First, we set  $u_{0/1} = 0$ ,  $u_{1/0} = 1$ . In general, for  $t \in \mathbf{Q} \cap (0, \infty)$ ,  $u_t$  is defined by

$$u_t \equiv m_s/m_r \pmod{m_t}. \quad (6)$$

Then  $u_t$  depends only on  $t$  but not the triple  $(r, t, s)$  since for the neighboring Farey triples  $(r, s', t)$  and  $(t, r', s)$  as shown in Figure 3 we have

$$m_r/m_{s'} \equiv m_s/m_r \equiv m_{r'}/m_s \pmod{m_t} \quad (7)$$

which in turn follows from

$$m_s m_{s'} = m_r^2 + m_t^2 \quad \text{and} \quad m_r m_{r'} = m_t^2 + m_s^2.$$

Now since  $0 \equiv m_r^2 + m_s^2 \equiv m_r^2(1 + u_t^2) \pmod{m_t}$  and  $\gcd(m_r, m_t) = 1$ , we have from (6)

$$u_t^2 + 1 \equiv 0 \pmod{m_t}. \quad (8)$$

Furthermore, we have

**Lemma 2.** *The ratio  $u_t/m_t$  is strictly increasing with respect to  $t \in \hat{\mathbf{Q}} \cap [0, \infty]$ . In particular,  $0 \leq u_t \leq m_t/2$ , with strict inequalities for  $t \neq \frac{0}{1}, \frac{1}{0}$ .*

In fact, Lemma 2 follows from the following

**Lemma 3.** *For every Farey triple  $(r, t, s)$  in  $\hat{\mathbf{Q}} \cap [0, \infty]$  with  $r < t < s$ ,*

$$\frac{u_t}{m_t} - \frac{u_r}{m_r} = \frac{m_s}{m_r m_t} \quad \text{and} \quad \frac{u_s}{m_s} - \frac{u_t}{m_t} = \frac{m_r}{m_t m_s}, \quad (9)$$

which are equivalent respectively to

$$u_t m_r - u_r m_t = m_s \quad \text{and} \quad u_s m_t - u_t m_s = m_r. \quad (10)$$

**PROOF.** We prove it by induction on the Farey levels of the rationals involved. The conclusion is easily checked to be true for the Farey triple  $(\frac{0}{1}, \frac{1}{1}, \frac{1}{0})$ . Now suppose that (10) holds for all Farey triples  $(r, t, s)$  in  $\hat{\mathbf{Q}} \cap [0, \infty]$  with  $r < t < s$  and with the Farey level of  $t$  not exceeding  $n \geq 1$ . In particular, this implies that  $0 \leq u_r/m_r < u_t/m_t < u_s/m_s \leq 1/2$ .

Then we only need to show that (10) also holds for the Farey triples  $(r, s', t)$  and  $(t, r', s)$  as shown in Figure 3. Since the proofs for the two cases are entirely similar, we prove it for the case  $(r, s', t)$  only, that is, we show that

$$u_{s'} m_r - u_r m_{s'} = m_t \quad \text{and} \quad u_t m_{s'} - u_{s'} m_t = m_r. \quad (11)$$

For this, we first see from (10) and  $m_r^2 + m_t^2 = m_{s'} m_s$  that

$$u := \frac{m_t + u_r m_{s'}}{m_r} = \frac{u_t m_{s'} - m_r}{m_t}. \quad (12)$$

Note that  $0 < u/m_{s'} < u_t/m_t < 1/2$  and, by (7),  $u$  is an integer. Hence (11) holds with  $u_{s'}$  replaced by  $u$ . But this in turn implies that  $u \equiv m_t/m_r \pmod{m_{s'}}$ , and hence  $u = u_{s'}$  by the definition of  $u_{s'}$ . This proves Lemma 3.  $\square$

**Remark.** The inequalities in (10) first appeared in [9, p.602], though they were contained essentially but implicitly in [13]. The result of Lemma 2 was stated and proved by Remak in [15]. In later part of this paper (see §5.2), Lemma 3 will also be obtained in a nice way as a corollary of the properties (see Proposition 7) of the so-called Markoff matrices which are interesting in their own right.

**2.4. Slope form of the Unicity Conjecture.** In terms of slopes, we may rephrase the Unicity Conjecture as

THE UNICITY CONJECTURE (SLOPE FORM). *The Markoff numbers  $m_t$ ,  $t \in \hat{\mathbf{Q}} \cap [0, \infty]$  are all distinct.*

### 3. PROOF OF THEOREM 1

We are now ready to give a very elementary proof for Theorem 1, using Lemma 2 and the following simple lemma whose proof can be found in [20].

**Lemma 4.** *Suppose  $m = p^n$  or  $2p^n$  for an odd prime  $p$  and an integer  $n \geq 1$ . Then, for any integer  $l$  coprime to  $m$ , the binomial congruence equation*

$$x^2 + l \equiv 0 \pmod{m} \quad (13)$$

*has at most one integer solution  $x$  with  $0 < x < m/2$ .*

PROOF OF THEOREM 1. Suppose there exist slopes  $t, t^* \in \hat{\mathbf{Q}} \cap [0, \infty]$  such that

$$m_t = m_{t^*} = p^n \text{ or } 2p^n$$

for an odd prime  $p$  and an integer  $n \geq 1$ . By (8) and its analog for  $u_{t^*}$ , Lemma 4 applies to give  $u_t = u_{t^*}$ . Then  $t = t^*$  by Lemma 2. This proves Theorem 1.  $\square$

**Remark.** The reader who is interested in merely the proof of Theorem 1 may well exit here. The rest of this paper is devoted to a discussion of the so-called Markoff matrices, which (with the exception of §5.2) constitutes the main body of an earlier version of this paper and can be used to prove our earlier results in a nice way.

### 4. MARKOFF MATRICES

It is Harvey Cohn [6] who first noticed the relationship of Markoff equation (1) and one of Fricke's trace identities, (16) below, for matrices in  $\mathrm{SL}(2, \mathbf{C})$ . This gives us a nice way to generate the Markoff numbers using the so-called Markoff matrices and hence to reformulate the Unicity Conjecture.

**4.1. Fricke's Trace identities.** In this subsection we derive some of Fricke's trace identities as needed.

**Proposition 5.** *If  $X, Y \in \mathrm{SL}(2, \mathbf{C})$  then*

$$\mathrm{tr}(XY) + \mathrm{tr}(XY^{-1}) = \mathrm{tr}(X) \mathrm{tr}(Y); \quad (14)$$

$$\mathrm{tr}^2(X) + \mathrm{tr}^2(Y) + \mathrm{tr}^2(XY) - \mathrm{tr}(X) \mathrm{tr}(Y) \mathrm{tr}(XY) = 2 + \mathrm{tr}(XYX^{-1}Y^{-1}). \quad (15)$$

*In particular, if  $X, Y \in \mathrm{SL}(2, \mathbf{C})$  satisfy  $\mathrm{tr}(XYX^{-1}Y^{-1}) = -2$  then*

$$\mathrm{tr}^2(X) + \mathrm{tr}^2(Y) + \mathrm{tr}^2(XY) = \mathrm{tr}(X) \mathrm{tr}(Y) \mathrm{tr}(XY). \quad (16)$$

PROOF. These identities can be verified easily by straightforward calculations. Here, however, we include a simpler derivation as presented in, for instance, [10] (see also [14]), which not only enables us to avoid tedious calculations but also would lead us to the rediscovery of the identities.

First, note that if  $Y = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  then  $Y^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ . Hence  $\mathrm{tr}(Y) = \mathrm{tr}(Y^{-1})$  and  $Y + Y^{-1} = \mathrm{tr}(Y) I$ , where  $I$  denotes the identity matrix. Then left-multiplying the latter equality by  $X$  gives

$$XY + XY^{-1} = X \mathrm{tr}(Y). \quad (17)$$



Taking traces on both sides of (17), we obtain identity (14). As a special case, we take  $X = Y$  in (14) to get

$$\mathrm{tr}(Y^2) = \mathrm{tr}^2(Y) - 2. \quad (18)$$

Finally, by making use of identity (14) repeatedly, we can calculate  $\mathrm{tr}(XYX^{-1}Y^{-1})$  and thus obtain (15) easily as follows:

$$\begin{aligned} \mathrm{tr}(XYX^{-1}Y^{-1}) &= \mathrm{tr}(X)\mathrm{tr}(YX^{-1}Y^{-1}) - \mathrm{tr}(XYXY^{-1}) \\ &= \mathrm{tr}^2(X) - [\mathrm{tr}(XY)\mathrm{tr}(XY^{-1}) - \mathrm{tr}(XYXY^{-1})] \\ &= \mathrm{tr}^2(X) - \mathrm{tr}(XY)[\mathrm{tr}(X)\mathrm{tr}(Y) - \mathrm{tr}(XY)] + \mathrm{tr}(Y^2) \\ &= \mathrm{tr}^2(X) - \mathrm{tr}(X)\mathrm{tr}(Y)\mathrm{tr}(XY) + \mathrm{tr}^2(XY) + \mathrm{tr}^2(Y) - 2. \end{aligned}$$

This proves Proposition 5.  $\square$

**Remark.** Many other trace identities of Fricke for matrices in  $\mathrm{SL}(2, \mathbf{C})$ , though shall not be needed in this paper, have been explored in [10] in details.

**4.2. Markoff matrices.** Following Cohn [6] but with a different choice, we associate a matrix in  $\mathrm{SL}(2, \mathbf{Z})$  to each slope  $t \in \hat{\mathbf{Q}} \cap [0, \infty]$  as follows. Initially, we set

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \quad (19)$$

and define

$$M_{\frac{0}{1}} = A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad M_{\frac{1}{0}} = AB = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}. \quad (20)$$

In general, for a Farey pair  $r, s \in \hat{\mathbf{Q}} \cup [0, \infty]$  with  $r < s$ , we set

$$M_{r \oplus s} = M_r M_s (\neq M_s M_r). \quad (21)$$

Thus we have defined for every  $t \in \hat{\mathbf{Q}} \cup [0, \infty]$  a *Markoff matrix*,  $M_t \in \mathrm{SL}(2, \mathbf{Z})$ , with positive elements. As a few more examples, one finds

$$\begin{aligned} M_{\frac{1}{2}} &= \begin{pmatrix} 21 & 29 \\ 13 & 18 \end{pmatrix}, \quad M_{\frac{1}{1}} = \begin{pmatrix} 8 & 11 \\ 5 & 7 \end{pmatrix}, \quad M_{\frac{2}{1}} = \begin{pmatrix} 46 & 65 \\ 29 & 41 \end{pmatrix}; \\ M_{\frac{1}{3}} &= \begin{pmatrix} 55 & 76 \\ 34 & 47 \end{pmatrix}, \quad M_{\frac{2}{3}} = \begin{pmatrix} 313 & 434 \\ 194 & 269 \end{pmatrix}, \quad M_{\frac{3}{2}} = \begin{pmatrix} 687 & 971 \\ 433 & 612 \end{pmatrix}, \quad M_{\frac{3}{1}} = \begin{pmatrix} 268 & 379 \\ 169 & 239 \end{pmatrix}. \end{aligned}$$

It is easy to observe that the trace of  $M_t$  equals 3 times the  $(2, 1)$ -element; for proof, see Proposition 7(iii), §5. Thus we may write for  $t \in \hat{\mathbf{Q}} \cap [0, \infty]$

$$m_t := \mathrm{tr}(M_t)/3. \quad (22)$$

Recall from §2.2 that by a Farey triple  $(r, t, s)$  in  $\hat{\mathbf{Q}} \cap [0, \infty]$  with  $r < t < s$  we mean that  $r, s \in \hat{\mathbf{Q}} \cap [0, \infty]$  are a Farey pair and that  $t = r \oplus s$ .

**Proposition 6.** *For every Farey triple  $(r, t, s)$  in  $\hat{\mathbf{Q}} \cap [0, \infty]$ ,  $(m_r, m_t, m_s)$  is a Markoff triple.*

**PROOF.** This follows from a simple application of identity (16) with  $X = M_r$  and  $Y = M_s$ . To apply (16), we need to verify that

$$\mathrm{tr}(M_r M_s M_r^{-1} M_s^{-1}) = -2 \quad (23)$$

for every pair of Farey neighbor  $r, s \in \hat{\mathbf{Q}} \cap [0, \infty]$  with  $r < s$ . Indeed, since

$$\text{tr}(M_r M_t M_r^{-1} M_t^{-1}) = \text{tr}(M_r M_s M_r^{-1} M_s^{-1}) = \text{tr}(M_t M_s M_t^{-1} M_s^{-1}),$$

it suffices to check (23) for the initial pair  $(r, s) = (\frac{0}{1}, \frac{1}{0})$ . This is true because

$$\text{tr}\left(M_{\frac{0}{1}} M_{\frac{1}{0}} M_{\frac{0}{1}}^{-1} M_{\frac{1}{0}}^{-1}\right) = \text{tr}\begin{pmatrix} -7 & 6 \\ -6 & 5 \end{pmatrix} = -2.$$

Since  $\text{tr} M_r = 3m_r$  etc., we obtain from (16) that

$$(3m_r)^2 + (3m_s)^2 + (3m_t)^2 = (3m_r)(3m_s)(3m_t).$$

This shows that  $(m_r, m_t, m_s)$  is a Markoff triple.  $\square$

**4.3. Matrix form of the Unicity Conjecture.** In terms of Markoff matrices defined above, we may rephrase the Unicity Conjecture as:

**THE UNICITY CONJECTURE (MATRIX FORM).** *The traces of Markoff matrices  $M_r$ ,  $r \in \hat{\mathbf{Q}} \cap [0, \infty]$  are all distinct.*

## 5. PROPERTIES OF MARKOFF MATRICES

The Markoff matrices defined in §4 possess certain nice properties which can be easily observed by inspecting just a few examples.

**5.1. Elements of a single Markoff matrix.** In a Markoff matrix, we have

**Proposition 7.** *For  $t \in \hat{\mathbf{Q}} \cap [0, \infty]$ , let  $M_t = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  be the Markoff matrix defined above. Then (i)  $c \leq d \leq a \leq b$ ; (ii)  $3a \geq 2b$ ,  $3c \geq 2d$ ; and (iii)  $a + d = 3c$ . Moreover, the inequalities in (i) and (ii) are all strict when  $t \neq 0, \infty$ .*

**PROOF.** We prove (i)–(iii) by induction on the Farey level of  $t$ . The conclusions (i)–(iii) are readily seen to be true for  $t \in \hat{\mathbf{Q}} \cap [0, \infty]$  of Farey level up to 1, that is, for  $r = \frac{0}{1}, \frac{1}{1}, \frac{1}{0}$ . Now suppose  $t \in \hat{\mathbf{Q}} \cap [0, \infty]$  has Farey level at least 2. As pointed out in §2.2, there exists a unique Farey pair  $r, s \in \hat{\mathbf{Q}} \cap [0, \infty]$  with  $r < s$ , such that  $t = r \oplus s$ . In particular,  $r$  and  $s$  have smaller Farey levels. Let

$$M_r = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad M_s = \begin{pmatrix} x & y \\ z & w \end{pmatrix}. \quad (24)$$

Then, by definition,

$$M_t = M_r M_s = \begin{pmatrix} ax + bz & ay + bw \\ cx + dz & cy + dw \end{pmatrix}. \quad (25)$$

To complete the inductive step, we proceed to prove (ii), (iii) and (i) in this order.

**PROOF OF (II) FOR THE INDUCTIVE STEP:** It suffices to observe that

$$\frac{y}{x} < \frac{ay + bw}{ax + bz} < \frac{cy + dw}{cx + dz} < \frac{w}{z} \leq \frac{3}{2}. \quad (26)$$

**PROOF OF (III) FOR THE INDUCTIVE STEP:** We need to show

$$(ax + bz) + (cy + dw) = 3(cx + dz). \quad (27)$$

The inductive hypothesis gives

$$a + d = 3c, \quad x + w = 3z. \quad (28)$$

Thus (27) is equivalent to

$$2dx = bz + cy. \quad (29)$$

There are two possibilities: the denominator (or numerator) of  $r$  is less or greater than that of  $s$ . Accordingly, we have  $s = r \oplus t'$  or  $r = t' \oplus s$ , where  $t' \in \hat{\mathbf{Q}} \cap [0, \infty]$  has Farey level lower than the maximum of those of  $r$  and  $s$ . In the case where  $s = r \oplus t'$  we have

$$M_{t'} = M_r^{-1} M_s = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} dx - bz & dy - bw \\ -cx + az & -cy + aw \end{pmatrix}. \quad (30)$$

Now the inductive hypothesis on  $M_{t'}$  yields

$$(dx - bz) + (-cy + aw) = 3(-cx + az) \quad (31)$$

which is, by (28), equivalent to (29). The proof for the other case is entirely similar.

**PROOF OF (i) FOR THE INDUCTIVE STEP:** The first and the last of the three inequalities in (i), that is,

$$ax + bz < ay + bw \quad \text{and} \quad cx + dz < cy + dw$$

follow easily from the inductive hypothesis  $x \leq y, z \leq w$ , of which at least one inequality is strict. It remains to prove

$$ax + bz > cy + dw. \quad (32)$$

By (27), this is equivalent to

$$3(cx + dz) > 2(cy + dw)$$

which is true since we have from the inductive hypothesis that  $3x \geq 2y$  and  $3z \geq 2w$ , and at least one of these two inequalities is strict.

This finishes the inductive step as well as the proof of Proposition 7.  $\square$

**5.2. Alternative proof of Lemma 3.** By Proposition 7, every Markoff matrix  $M_t$ ,  $t \in \hat{\mathbf{Q}} \cap [0, \infty]$  is of the form

$$M_t = \begin{pmatrix} 2m_t - u & * \\ m_t & m_t + u \end{pmatrix},$$

where  $0 \leq u \leq m_t/2$ . Now  $\det M_t = 1$  implies that  $u^2 + 1 \equiv 0 \pmod{m_t}$ . By the definition of  $u_t$ , we have  $u = u_t$ . Thus we obtain

**Proposition 8.** *For every  $t \in \hat{\mathbf{Q}} \cap [0, \infty]$ , the Markoff matrix*

$$M_t = \begin{pmatrix} 2m_t - u_t & 2m_t + u_t - v_t \\ m_t & m_t + u_t \end{pmatrix}. \quad (33)$$

Using (33), we can give an alternative proof of Lemma 3.

**ALTERNATIVE PROOF OF LEMMA 3.** We obtain from  $M_r M_s = M_t$  that

$$M_s = M_r^{-1} M_t = \begin{pmatrix} * & * \\ u_t m_r - u_r m_t & * \end{pmatrix}.$$

Equating the (2, 1)-elements then gives the first equality in (10). The second equality in (10) follows similarly from  $M_r = M_t M_s^{-1}$ .  $\square$

**5.3. Monotonicity of the index of a Markoff matrix.** It is convenient to introduce an index

$$\varrho(M_t) := \frac{a}{c} \quad (34)$$

for every Markoff matrix  $M_t = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ , where  $r \in \hat{\mathbf{Q}} \cap [0, \infty]$ .

We then have the following monotonicity of the index of a Markoff matrix with respect to its slope. This follows readily from Lemma 2 and (33). However, we choose to include a direct simple proof which in turn gives an alternative proof of Lemma 2.

**Proposition 9.** *Suppose  $t_1, t_2 \in \hat{\mathbf{Q}} \cap [0, \infty]$  where  $t_1 < t_2$ . Then  $\varrho(M_{t_1}) > \varrho(M_{t_2})$ .*

**PROOF.** We proceed to prove this proposition by induction on the maximum of the Farey levels of  $t_1$  and  $t_2$ . First, the conclusion is true for  $t_1, t_2$  both having Farey level 0, that is,  $t_1 = \frac{0}{1}$  and  $t_2 = \frac{1}{0}$ , since  $\varrho(M_{\frac{0}{1}}) = 2/1$  and  $\varrho(M_{\frac{1}{0}}) = 3/2$ .

By the process of constructing all the rationals in  $[0, \infty]$  by recursively making Farey sums as described in §2.2, we only need to prove the conclusion locally, that is, suppose it is true for a Farey pair  $r, s \in \hat{\mathbf{Q}} \cap [0, \infty]$  with  $r < s$  and show

$$\varrho(M_r) > \varrho(M_t) > \varrho(M_s), \quad (35)$$

where we have written  $t := r \oplus s$ . Suppose  $M_r, M_s$  are given by (24). Then  $M_t$  is given by (25). By our inductive hypothesis,  $\varrho(M_r) > \varrho(M_s)$ , that is,  $a/c > x/z$ . Then (35) is equivalent to

$$\frac{a}{c} > \frac{ax + bz}{cx + dz} > \frac{x}{z}. \quad (36)$$

The first inequality in (36) follows easily from the fact that  $a/c > b/d$  (since  $ad - bc = 1$ ). The second is equivalent, by Proposition 7 (iii), to the inequality

$$\frac{cy + dw}{cx + dz} < \frac{w}{z},$$

which is true since  $y/x < w/z$  (as  $xw - yz = 1$ ). This finishes the inductive step as well as the proof of Proposition 9.  $\square$

As an immediate corollary of Proposition 9, we obtain that

**Proposition 10.** *The Markoff matrices  $M_t, t \in \hat{\mathbf{Q}} \cap [0, \infty]$  are all distinct.*

**Remark.** There are other choices in the definition of Markoff matrices, such as

$$M_{\frac{0}{1}} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad M_{\frac{1}{0}} = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}. \quad (37)$$

For this choice we have for all  $t \in \hat{\mathbf{Q}} \cap [0, \infty]$

$$M_t = \begin{pmatrix} 2m_t + u_t & m_t \\ 2m_t - u_t - v_t & m_t - u_t \end{pmatrix}. \quad (38)$$

## 6. FURTHER REMARKS

In this section we make further remarks to give a geometric explanation of the Markoff numbers and related numbers.

**6.1. Once-cusped hyperbolic torus.** It is known from Cohn's work [6] that the Markoff numbers correspond to the simple closed geodesics on a special hyperbolic torus with a single cusp. (See [18] an exposition of the background.) Specifically, let  $A, B \in \mathrm{SL}(2, \mathbf{R})$  be given as in §4.2 and let  $\langle A, B \rangle \subset \mathrm{SL}(2, \mathbf{R})$  be the subgroup generated by  $A$  and  $B$ . Then  $\langle A, B \rangle$  is a Fuchsian group and  $T := \mathbb{H}^2 / \langle A, B \rangle$  is once-cusped hyperbolic torus, where  $\mathbb{H}^2$  is the upper half-plane model of the hyperbolic plane. Note that the axes of the Möbius transformations  $A$ ,  $B$  and  $AB$  project onto simple closed curves on  $T$ . Assign to these three simple closed curves on  $T$  the slopes  $\frac{0}{1}$ ,  $\frac{-1}{1}$  and  $\frac{1}{0}$  respectively, and consider all the simple closed curves  $\gamma_t$  on  $T$  of slopes  $t \in \hat{\mathbf{Q}} \cap [0, \infty]$ . Let the hyperbolic length of  $\gamma_t$  be  $l_t$ . Then we have the relation

$$3m_t = 2 \cosh(l_t/2).$$

Hence the Unicity Conjecture is actually a conjecture about the uniqueness of lengths of certain simple closed geodesics on the specific hyperbolic torus  $T$ .

**6.2. McShane identity.** For a Farey triple  $(r, t, s)$  in  $\hat{\mathbf{Q}} \cap [0, \infty]$  with  $r < t < s$ , the quantity  $\frac{m_{t'}}{m_r m_s}$  (where  $m_{t'} = 3m_r m_s - m_t$ ) appeared in (9) has nice geometric meanings. In particular, it leads naturally to the interesting McShane identity; see [2] (Theorem 3 and Proposition 3.13 there) for details.

**6.3. Exceptional vector bundles on  $\mathbf{CP}^2$ .** In an unexpected way, the Markoff numbers also appear as the ranks of the exceptional vector bundles on  $\mathbf{CP}^2$ , as explained by Rudakov [16]. In this context, the quantity  $u/m$  is the “slopes” the corresponding vector bundles, with  $u$  the first Chern class (which is an integer in this case).

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DEPARTMENT OF MATHEMATICS, YANGZHOU UNIVERSITY, YANGZHOU 225002, CHINA  
*E-mail address:* `yingzhang@yzu.edu.cn`

CURRENT ADDRESS: IMPA, ESTRADA DONA CASTORINA 110, 22460 RIO DE JANEIRO, BRAZIL  
*E-mail address:* `yiing@impa.br`